Trigonometry 2

1. Solve $\sin\theta\csc\frac{\theta}{2}-\cot\frac{\theta}{2}=3\left(1-2\sin\frac{\theta}{2}\right)$ for $-360°\le\theta\le 360°$.

$\sin\theta\csc\frac{\theta}{2}-\cot\frac{\theta}{2}=3\left(1-2\sin\frac{\theta}{2}\right)\Longrightarrow\frac{\sin\theta}{\sin\frac{\theta}{2}}-\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=3\left(1-2\sin\frac{\theta}{2}\right)$

$\Longrightarrow\sin\theta-\cos\frac{\theta}{2}=3\sin\frac{\theta}{2}\left(1-2\sin\frac{\theta}{2}\right)\Longrightarrow 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}-\cos\frac{\theta}{2}=3\sin\frac{\theta}{2}\left(1-2\sin\frac{\theta}{2}\right)$

$\Longrightarrow\cos\frac{\theta}{2}\left(2\sin\frac{\theta}{2}-1\right)+3\sin\frac{\theta}{2}\left(2\sin\frac{\theta}{2}-1\right)=0\Longrightarrow\left(3\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)\left(2\sin\frac{\theta}{2}-1\right)=0$

$\Longrightarrow 3\sin\frac{\theta}{2}+\cos\frac{\theta}{2}=0\ \ or\ \ 2\sin\frac{\theta}{2}-1=0\Longrightarrow\tan\frac{\theta}{2}=-\frac{1}{3}$ or $\sin\frac{\theta}{2}=\frac{1}{2}$

For $\tan\frac{\theta}{2}=-\frac{1}{3}$, $\theta=360°\,n-2(18.434948822922)$, where n is an integer.

Since $-360°\le\theta\le 360°$, $\theta=-36.869897645844°\ \ or\ 323.130102354156°$

For $\sin\frac{\theta}{2}=\frac{1}{2}$, $\theta=720°\,n+60°\ or\ \theta=720°\,n+300°$, where n is an integer.

Since $-360°\le\theta\le 360°$, $\theta=60°\ \ or\ \ 300°$.

$\theta=\mathbf{-36.869897645844°}$, $\mathbf{60°}$, $\mathbf{300°}$, $\mathbf{323.130102354156°}$

2. Show that for all values of θ ,

(a) $cos\ \theta+cos\ \left(\theta+\frac{2}{3}\pi\right)+cos\ \left(\theta+\frac{4}{3}\pi\right)=0$

(b) $sin^2\ \theta+sin^2\ \left(\theta+\frac{2}{3}\pi\right)+sin^2\ \left(\theta+\frac{4}{3}\pi\right)=\frac{3}{2}$

(c) $cos^3\ \theta+cos^3\ \left(\theta+\frac{2}{3}\pi\right)+cos^3\ \left(\theta+\frac{4}{3}\pi\right)=\frac{3}{4}\cos 3\theta$.

(a) $cos\ \theta+cos\ \left(\theta+\frac{2}{3}\pi\right)+cos\ \left(\theta+\frac{4}{3}\pi\right)=cos\ \theta-2\ cos\ \frac{\left(\theta+\frac{4}{3}\pi\right)+\left(\theta+\frac{2}{3}\pi\right)}{2}cos\ \frac{\left(\theta+\frac{4}{3}\pi\right)-\left(\theta+\frac{2}{3}\pi\right)}{2}$

$=cos\ \theta-2\ cos\ (\theta+2\pi)cos\ \frac{\pi}{6}=cos\ \theta+2\ cos\ \theta\ \left(\frac{1}{2}\right)=0$

(b) $sin^2\ \theta+sin^2\ \left(\theta+\frac{2}{3}\pi\right)+sin^2\ \left(\theta+\frac{4}{3}\pi\right)$

$=\frac{1}{2}[1-\cos 2\theta]+\frac{1}{2}\left[1-\cos 2\left(\theta+\frac{2}{3}\pi\right)\right]+\frac{1}{2}\left[1-\cos 2\left(\theta+\frac{4}{3}\pi\right)\right]$

$=\frac{3}{2}-\frac{1}{2}\left[\cos 2\theta+\cos\left(2\theta+\frac{4}{3}\pi\right)+\cos\left(2\theta+\frac{8}{3}\pi\right)\right]$

$=\frac{3}{2}-\frac{1}{2}\left[\cos 2\theta+\cos\left(2\theta+\frac{4}{3}\pi\right)+\cos\left(2\theta+\frac{8}{3}\pi-2\pi\right)\right]$

$= \frac{3}{2} - \frac{1}{2}\left[\cos 2\theta + \cos\left(2\theta + \frac{2}{3}\pi\right) + \cos\left(2\theta + \frac{4}{3}\pi\right)\right] = \frac{3}{2} - \frac{1}{2}[0]$, by (a), replace θ by 2θ

$= \frac{3}{2}$

(c) Consider:

$4\left[cos^3\,\theta + cos^3\,\left(\theta + \frac{2}{3}\pi\right) + cos^3\,\left(\theta + \frac{4}{3}\pi\right)\right]$, and using (a)

$= 4\left[cos^3\,\theta + cos^3\,\left(\theta + \frac{2}{3}\pi\right) + cos^3\,\left(\theta + \frac{4}{3}\pi\right)\right] - 3\left[cos\,\theta + cos\,\left(\theta + \frac{2}{3}\pi\right) + cos\,\left(\theta + \frac{4}{3}\pi\right)\right]$

$= [4cos^3\,\theta - 3cos\,\theta] + \left[4cos^3\,\left(\theta + \frac{2}{3}\pi\right) - 3cos\,\left(\theta + \frac{2}{3}\pi\right)\right] + \left[4cos^3\,\left(\theta + \frac{4}{3}\pi\right) - 3cos\,\left(\theta + \frac{4}{3}\pi\right)\right]$

$= \cos 3\theta + \ cos\,3\left(\theta + \frac{2}{3}\pi\right) + \ cos\,3\left(\theta + \frac{4}{3}\pi\right)$

$= \cos 3\theta + \ cos\,(3\theta + 2\pi) + \ cos\,(3\theta + 2\pi)$

$= \cos 3\theta + \cos 3\theta + \cos 3\theta$

$= 3\cos 3\theta$

Hence, $cos^3\,\theta + cos^3\,\left(\theta + \frac{2}{3}\pi\right) + cos^3\,\left(\theta + \frac{4}{3}\pi\right) = \frac{3}{4}\cos 3\theta$

3. Prove $-\frac{1}{\sec 2x} \equiv \frac{2(sin^3\,x - cos^3\,x)}{\sin x + \cos x} + \frac{\cos 2x}{(\sin x + \cos x)^2}$.

R.H.S. $= \frac{2(\sin x - \cos x)(sin^2\,x + \sin x\cos x + cos^2\,x)}{\sin x + \cos x} + \frac{\cos 2x}{(\sin x + \cos x)^2}$

$= \frac{2(\sin x + \cos x)(\sin x - \cos x)(1 + \sin x\cos x)}{(\sin x + \cos x)^2} + \frac{\cos 2x}{(\sin x + \cos x)^2}$

$= \frac{2(sin^2\,x - cos^2\,x)(1 + \sin x\cos x)}{(\sin x + \cos x)^2} + \frac{\cos 2x}{(\sin x + \cos x)^2}$

$= \frac{-2\cos 2x(1 + \sin x\cos x)}{(\sin x + \cos x)^2} + \frac{\cos 2x}{(\sin x + \cos x)^2}$

$= \frac{-\cos 2x}{(\sin x + \cos x)^2}[2(1 + \sin x\cos x) - 1]$

$= \frac{-\cos 2x}{(\sin x + \cos x)^2}[1 + 2\sin x\cos x]$

$= \frac{-\cos 2x}{(\sin x + \cos x)^2}(\sin x + \cos x)^2$

$= -\cos 2x$

$=$ L.H.S.

4. Given that $y = \frac{5 + 4\cos\theta}{1 + \sin\theta}$. Use the substitution $t = \tan\frac{\theta}{2}$,show that $y = \frac{9 + t^2}{1 + 2t + t^2}$.

Hence, or otherwise, prove that $y \geq \frac{9}{10}$.

Let $t = \tan\frac{\theta}{2}$, $\quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}} = \frac{2\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}+1} = \frac{2t}{1+t^2}$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}} = \frac{1-\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}+1} = \frac{1-t^2}{1+t^2}$$

$y = \frac{5+4\cos\theta}{1+\sin\theta} = \frac{5+4\left(\frac{1-t^2}{1+t^2}\right)}{1+\frac{2t}{1+t^2}} = \frac{5(1+t^2)+4(1-t^2)}{1+t^2+2t} = \frac{9+t^2}{1+2t+t^2} \Longrightarrow y(1+2t+t^2) = 9+t^2$

$(y-1)t^2 + (2y)t + y - 9 = 0$

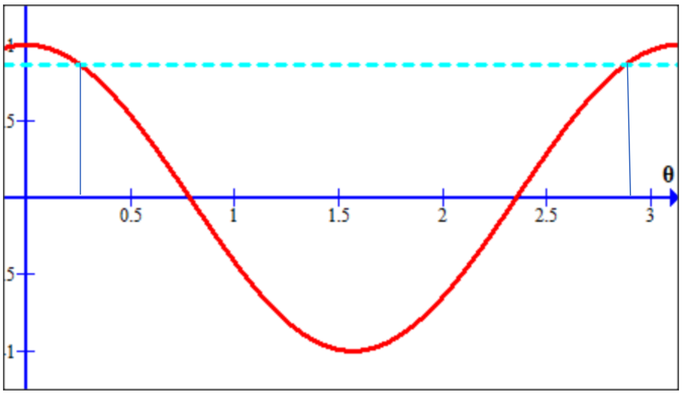
Since t is real , $\Delta = (2y)^2 - 4(y-1)(y-9) \geq 0 \Longrightarrow y \geq \frac{9}{10}$

5. Sketch a graph $y = \cos 2\theta$ in the range $0 \leq \theta \leq \pi$. Hence, find the set of values of θ, where $0 \leq \theta \leq \pi$, satisfying the inequality $4\sin^2\theta \geq 2-\sqrt{3}$.

$4\sin^2\theta \geq 2-\sqrt{3} \Longrightarrow 2(1-\cos 2\theta) \geq 2-\sqrt{3}$

$\Longrightarrow 1-\cos 2\theta \geq \frac{2-\sqrt{3}}{2} \Longrightarrow \cos 2\theta \leq 1-\frac{2-\sqrt{3}}{2}$

$\Longrightarrow \cos 2\theta \leq \frac{\sqrt{3}}{2}$



For the equation, $\cos 2\theta = \frac{\sqrt{3}}{2}$

We get $2\theta = \frac{\pi}{6}, \frac{11\pi}{6}$ $\quad$ and $= \frac{\pi}{12}, \frac{11\pi}{12}$.

By drawing the horizontal line $y = \frac{\sqrt{3}}{2}$, we get the solution for $\cos 2\theta \leq \frac{\sqrt{3}}{2}$

and hence $4\sin^2\theta \geq 2-\sqrt{3}$ is $\quad \frac{\pi}{12} \leq \theta \leq \frac{11\pi}{12}$.

6. Solve $\sin 2\theta - 2\sin\theta = 1$ for $\theta = 0° \; to \; 360°$.

$\sin 2\theta - 2\sin\theta = 1 \Longrightarrow 2\sin\theta\cos\theta - 2\sin\theta = 1$

Put $t = \tan\frac{\theta}{2}$ then $\sin\theta = \frac{2t}{1+t^2}, \cos\theta = \frac{1-t^2}{1+t^2}$

$2\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) = 1$

$4t(1-t^2) - 4t(1+t^2) = (1+t^2)^2$

$(1+t^2)^2 + 4t(1+t^2) - 4t(1-t^2) = 0$

$t^4 + 8t^3 + 2t^2 + 1 = 0$

Use numerical method such as Newton's method to solve for the real roots:

$t \approx -7.73943 \; or - 0.621711$

$\tan\frac{\theta}{2} \approx -7.73943 \; or - 0.621711$

$\theta \approx 194.72° \; or \; 296.26°$

7. If $f(r) = \cos 2r\theta$, simplify $f(r) - f(r-1)$. Use your result to find the sum of the first n terms of the series $\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots$.

$f(r) - f(r-1) = \cos 2r\theta - \cos 2(r-1)\theta = -\frac{1}{2} sin\frac{2r\theta + 2(r-1)\theta}{2} sin\frac{2r\theta - 2(r-1)\theta}{2}$

$= -\frac{1}{2}\sin(2r-1)\theta \sin\theta$

$\sum_{r=2}^{n}[f(r) - f(r-1)] = -\frac{1}{2}\sin\theta \sum_{\mathrm{r}=2}^{\mathrm{n}} \sin(2r-1)\theta$

$f(n) - f(1) = -\frac{1}{2}\sin\theta\,[\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots + \sin(2n-1)\theta]$

$\cos 2n\theta - \cos 2\theta = -\frac{1}{2}\sin\theta\,[\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots + \sin(2n-1)\theta]$

$\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots + \sin(2n-1)\theta = 2\left(\frac{\cos 2\theta - \cos 2n\theta}{\sin\theta}\right)$

Sum of the first n terms of the series $\sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots$

$= \sin 3\theta + \sin 5\theta + \sin 7\,\theta + \cdots + \sin(2n-1)\theta + \sin(2n+1)\theta = 2\left[\frac{\cos 2\theta - \cos 2(n+1)\theta}{\sin\theta}\right]$

8. Solve the equation $cos^{-1}2x + sin^{-1}x = \frac{\pi}{3}$.

$cos[cos^{-1}2x + sin^{-1}x] = cos\frac{\pi}{3} \Longrightarrow cos[cos^{-1}2x + sin^{-1}x] = \frac{1}{2}$

$cos(cos^{-1}2x)cos(sin^{-1}x) - sin(cos^{-1}2x)sin(sin^{-1}x) = \frac{1}{2}$

By drawing right angled triangles, we have

$(2x)(\sqrt{1-x^2}) - (\sqrt{1-4x^2})(x) = \frac{1}{2} \Longrightarrow 4x\sqrt{1-x^2} - 2x\sqrt{1-4x^2} = 1 \ldots (1)$

$\left[4x\sqrt{1-x^2} - 2x\sqrt{1-4x^2}\right]\left[4x\sqrt{1-x^2} + 2x\sqrt{1-4x^2}\right] = 4x\sqrt{1-x^2} + 2x\sqrt{1-4x^2}$

$16x^2(1-x^2)-4x^2(1-4x^2)=4x\sqrt{1-x^2}+2x\sqrt{1-4x^2}$

Therefore $4x\sqrt{1-x^2}+2x\sqrt{1-4x^2}=12x^2 \ldots (2)$

$(1)+(2), 8x\sqrt{1-x^2}=12x^2+1$

Squaring, $64x^2(1-x^2)=144x^4+24x^2+1$

$208x^4-40x^2+1=0,$

Solving, $x^2=\frac{5+2\sqrt{3}}{52}$, using only the positive root.

$\therefore x=\sqrt{\frac{5+2\sqrt{3}}{52}}$

9. Prove that $\frac{2\sin 4\theta-\sin 6\theta-\sin 2\theta}{2\sin 4\theta+\sin 6\theta+\sin 2\theta}=\tan^2\theta$.

Hence, find the value of $\tan^2 15°$, leaving your answer in surd form.

$$\frac{2\sin 4\theta-\sin 6\theta-\sin 2\theta}{2\sin 4\theta+\sin 6\theta+\sin 2\theta}=\frac{2\sin 4\theta-2\sin\left(\frac{6\theta+2\theta}{2}\right)cos\left(\frac{6\theta-2\theta}{2}\right)}{2\sin 4\theta+2\sin\left(\frac{6\theta+2\theta}{2}\right)cos\left(\frac{6\theta-2\theta}{2}\right)}=\frac{2\sin 4\theta-2\sin 4\theta cos2\theta}{2\sin 4\theta+2\sin 4\theta cos2\theta}=\frac{1-cos2\theta}{1+cos2\theta}=\frac{2\sin^2\theta}{2\cos^2\theta}=\tan^2\theta$$

$$\tan^2 15°=\frac{2\sin 60°-\sin 90°-\sin 30°}{2\sin 60°+\sin 90°+\sin 30°}=\frac{2\left(\frac{\sqrt{3}}{2}\right)-1-\left(\frac{1}{2}\right)}{2\left(\frac{\sqrt{3}}{2}\right)+1+\left(\frac{1}{2}\right)}=\frac{2\sqrt{3}-2-1}{2\sqrt{3}+2+1}=\frac{2\sqrt{3}-3}{2\sqrt{3}+3}=\frac{\left(2\sqrt{3}-3\right)^2}{\left(2\sqrt{3}+3\right)\left(2\sqrt{3}-3\right)}$$

$$=\frac{12-12\sqrt{3}+9}{3}=\frac{21-12\sqrt{3}}{3}=\mathbf{7-4\sqrt{3}}$$

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